Lie discrete symmetries of lattice equations

2004 J. Phys. A: Math. Gen. 37 1711

(http://iopscience.iop.org/0305-4470/37/5/016)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 1711–1725 PII: S0305-4470(04)65751-9

Lie discrete symmetries of lattice equations

Decio Levi[1] **and Miguel A Rodríguez**

Departamento de Física Teórica II, Facultad de Físicas, Universidad Complutense, 28040-Madrid, Spain

E-mail: levi@fis.uniroma3.it and rodrigue@fis.ucm.es

Received 7 July 2003, in final form 10 September 2003
Published 19 January 2004
Online at stacks.iop.org/JPhysA/37/1711 (DOI: 10.1088/0305-4470/37/5/016)

Abstract

We extend two of the methods previously introduced to find discrete symmetries of differential equations to the case of difference and differential-difference equations. As an example of the application of the methods, we construct the discrete symmetries of the discrete Painlevé I equation and of the Toda lattice equation.

PACS numbers: 02.20.Sv, 02.30.Ks

1. Introduction

Symmetries have always played a very important role in the study of differential equations [1]. Lie symmetries were introduced by Sophus Lie as a tool to unify all solution techniques for ordinary differential equations. In particular they are useful to

- get special solutions by *symmetry reduction*.
- classify equations according to their symmetry.
- prove equivalence of equations under point transformations.

They have been extended with success to the case of differential-difference and difference-difference equations. As was shown in [2] the invariance of a differential-difference equation with respect to a shift of the lattice $\tilde{n} = n + N$ provides a reduction of the equation to a system of N ordinary differential equations. Moreover, discrete symmetries, i.e. symmetries associated with a discrete finite group, are very important in quantum mechanics. In this field one speaks of *parity, charge conjugation, rotations by* π, etc, and one uses discrete symmetries to provide selection rules.

Discrete symmetries are usually easy to guess but difficult to find in a systematic way. They can be obtained by considering the normalizer of the continuous Lie point symmetries of the equation. This is a well-known technique and can be found in many textbooks (see,

[1] Permanent address: Dipartimento di Fisica, Università Roma Tre and INFN-Sezione di Roma Tre, Via della Vasca Navale 84, 00146-Roma, Italy.

for instance, [6]). The application to the case of differential equations has been considered in all details by Hydon [7]. The normalizer can be obtained only in the case when a nontrivial non-commuting group of Lie point symmetries exists.

In 1996 Gaeta and Rodríguez [8] introduced a modification of the Lie technique which allows us to find discrete symmetries even when the Lie group is trivial. This technique, as we will see in section 3, depends on a starting ansatz and thus it does not provide a complete result.

Reid *et al* [3] solved the determining equations for the group transformations directly to get the discrete symmetries. In more recent works [4, 5] they simplified the work taking advantage of the infinitesimal results; however, the method requires extremely heavy calculations.

In section 2 we briefly summarize the steps necessary to obtain Lie symmetries for difference equations, while in section 3 we present two of the methods introduced to obtain discrete symmetries. As is evident from section 2 all the steps necessary to obtain the discrete Lie symmetries are not significantly different in the case of equations on the lattice from the continuous case. The only essential difference is in the prolongation formula. So, in section 4 we just present two examples of the derivation of discrete Lie symmetries for discrete equations. In section 5 one can find some concluding remarks.

2. Point symmetries of discrete equations

Let us briefly summarize the steps necessary to obtain Lie point symmetries for difference equations. More details can be found in [9, 10].

We will consider just the case of a scalar difference equation in two independent lattice variables as this case will cover all the examples considered in section 4. The case of a differential-difference equation is obtained from the difference-difference case by carrying out the continuum limit in one variable.

A discrete equation of order N_i in a discrete variable $x^{(i)}_{n,m}$, $i = 1, 2$, is a functional relation between $N_i + 1$ points in the lattice of the variable $x^{(i)}_{n,m}$. To be able to solve the discrete equation apart from the functional relation, we need to know how the lattice points are defined. This implies that we need a further four equations compatible in the case of two independent variables and one for one independent lattice variable. Some of these equations may be trivial if the lattice is orthogonal and with constant spacing, but are still necessary to fix the symmetries and the solutions of the difference system.

So we have

$$\Delta\big(x^{(k)}_{n+j,m+i}, u_{n+j,m+i}\big) = 0 \tag{2.1}$$

$$E_a\big(x^{(k)}_{n+j,m+i}, u_{n+j,m+i}\big) = 0 \tag{2.2}$$

$$u_{n,m} = u\big(x^{(1)}_{n,m}, x^{(2)}_{n,m}\big) \qquad 1 \leqslant a \leqslant 4 \quad k = 1, 2 \quad -i_1 \leqslant i \leqslant i_2$$

$$-j_1 \leqslant j \leqslant j_2 \quad i_1, i_2, j_1, j_2 \in Z^{\geqslant 0}$$

where Δ is the difference equation, E_a are the equations determining the two independent lattice variables and n and m are some indices which characterize them. System (2.1), (2.2) must satisfy some obvious conditions so as to be able to calculate the variables in all the lattice plane.

A Lie point symmetry for equations (2.1), (2.2) is a point transformation

$$\tilde{x}^{(i)}_{n,m} = F^{(i)}_\lambda\big(x^{(k)}_{n,m}, u_{n,m}\big) \qquad \tilde{u}_{n,m} = G_\lambda\big(x^{(k)}_{n,m}, u_{n,m}\big) \qquad (i, k = 1, 2) \tag{2.3}$$

which leaves equations (2.1), (2.2) invariant. The symbol λ indicates the group parameters. As this transformation acts on the entire space where the independent and dependent variables are defined, the functions $F_\lambda^{(i)}$ and G_λ determine the transformation everywhere. We can introduce the corresponding infinitesimal transformations of coefficients $\xi^{(i)}\big(x_{n,m}^{(k)}, u_{n,m}\big)$, $\phi\big(x_{n,m}^{(k)}, u_{n,m}\big)$, and thus write the vector field

$$\hat{X}_{n,m} = \xi^{(i)}\big(x_{n,m}^{(k)}, u_{n,m}\big)\partial_{x_{n,m}^{(i)}} + \phi\big(x_{n,m}^{(k)}, u_{n,m}\big)\partial_{u_{n,m}} \tag{2.4}$$

and its prolongation

$$\mathrm{pr}\hat{X}_{n,m} = \sum_{j=-j_1}^{j_2}\sum_{l=-i_1}^{i_2} \hat{X}_{n+l,m+j}. \tag{2.5}$$

The invariance conditions, i.e. the necessary conditions which provide the symmetries for equations (2.1), (2.2), are given by

$$\mathrm{pr}\hat{X}_{n,m}\Delta = 0 \qquad \text{when} \quad (\Delta, E_a) = (0, 0) \tag{2.6}$$

$$\mathrm{pr}\hat{X}_{n,m}E_a = 0 \qquad \text{when} \quad (\Delta, E_a) = (0, 0). \tag{2.7}$$

Equations (2.6), (2.7) are a set of equations for $\xi^{(i)}$ and ϕ. In equations (2.6), (2.7) $\xi^{(i)}$ and ϕ will appear at various points of the lattice, which, when the equations $\Delta = 0$ and $E_a = 0$ have been taken into account, are all independent. So we have five functional equations for the functions $\xi^{(i)}$ and ϕ. The variables appearing in these five equations are all independent. These independent variables appear either explicitly in the equation or in the unknown infinitesimal coefficients. As the infinitesimal coefficients are analytic functions, we can convert the determining equations into a system of differential equations by differentiating them with respect to the independent variables. In such a way we get an overdetermined system of, in general, nonlinear partial differential equations. We solve the obtained equations and introduce their solution into the functional equations and solve them.

As an example of this procedure we will calculate the continuous symmetries of a discrete Painlevé I equation [11]. The discrete Painlevé I equation is given by

$$u_{n+1} + u_n + u_{n-1} = \frac{\alpha x_n + \beta}{u_n} + \gamma \tag{2.8}$$

where α, β and γ are arbitrary constants and $u_n = u(x_n)$. The equation which defines the lattice points can be written as

$$x_{n+1} - x_n = h \tag{2.9}$$

so that $x_n = hn + x_0$. With the choice $u_n = -1/2 + h^2 w(x)$, $\alpha = -h^4/2$, $\beta = -3/4$ and $\gamma = -3$, equation (2.8) reduces, in the continuous limit, to

$$w_{xx} = 6w^2 + x \tag{2.10}$$

which is the Painlevé I transcendent [12].

The infinitesimal symmetry generator is given by

$$\hat{X}_n = \xi_n(x_n, u_n)\partial_{x_n} + \phi_n(x_n, u_n)\partial_{u_n} \tag{2.11}$$

and its prolongation

$$\mathrm{pr}\hat{X}_n = \sum_{j=-1}^{1} \hat{X}_{n+j}. \tag{2.12}$$

Applying equation (2.12) to the lattice equation (2.9) we get

$$\xi_{n+1}(x_{n+1}, u_{n+1}) = \xi_n(x_n, u_n) \tag{2.13}$$

which, taking into account that u_n and u_{n+1} are independent variables, gives that ξ_n cannot depend on u_n and, as a function of n, must be a constant (cannot vary between different points of the lattice)

$$\xi_n = K_0. \tag{2.14}$$

In a similar way, by applying (2.12) to equation (2.8) we get

$$-\frac{K_0\alpha}{u_n} + \phi_{n+1}(x_{n+1}, u_{n+1}) + \phi_n(x_n, u_n) + \phi_{n-1}(x_{n-1}, u_{n-1}) + \frac{\alpha x_n + \beta}{u_n^2}\phi_n(x_n, u_n) = 0 \tag{2.15}$$

where

$$u_{n-1} = \frac{\alpha x_n + \beta}{u_n} + \gamma - u_{n+1} - u_n \tag{2.16}$$

and differentiating (2.15) with respect to u_n and u_{n+1}, one can prove that

$$\phi_n(x_n, u_n) = \phi_n^{(0)}(x_n) + u_n\phi_n^{(1)}(x_n) \tag{2.17}$$

where $\phi_n^{(0)}$ and $\phi_n^{(1)}$ satisfy the following equations

$$\phi_{n-1}^{(1)} = \phi_n^{(1)} \tag{2.18}$$

$$(\alpha x_n + \beta)\big(\phi_n^{(1)} + \phi_{n-1}^{(1)}\big) = \alpha K_0 \tag{2.19}$$

$$(\alpha x_n + \beta)\phi_n^{(0)} = 0 \tag{2.20}$$

$$\phi_{n+1}^{(0)} + \phi_n^{(0)} + \phi_{n-1}^{(0)} + \gamma\phi_{n-1}^{(1)} = 0. \tag{2.21}$$

According to the values of the parameters (α, β and γ) we have various possibilities:

- If $\alpha \neq 0$ then from equation (2.19) $K_0 = 0$ and $\phi_n^{(1)} = 0$. From equation (2.20) $\phi_n^{(0)} = 0$ and thus no symmetry is present.
- If $\alpha = 0$ but $\beta \neq 0$ then from equation (2.19), $\phi_n^{(1)} = 0$ and from equation (2.20), $\phi_n^{(0)} = 0$; so only $K_0 \neq 0$, i.e. only space translations are possible.
- If $\alpha = 0$, $\beta = 0$ equations (2.19), (2.20) are identically satisfied; equation (2.18) implies that $\phi_n^{(1)} = K_1$ and [13]

$$\phi_n^{(0)} = -\frac{\gamma K_1}{3} + K_2\left[\frac{1}{2}(\mathrm{i}\sqrt{3} - 1)\right]^n + K_3\left[-\frac{1}{2}(\mathrm{i}\sqrt{3} + 1)\right]^n. \tag{2.22}$$

So in this case now the linear equation has a four-dimensional symmetry group.

To end this section let us consider the case of symmetries of differential-difference equations. The transition from a difference-difference equation to a differential-difference equation is done by carrying out the continuum limit for a lattice variable, say $x_{n,m}^{(1)} = t_{n,m}$, when the distance between the points along this direction goes to zero and the lattice index goes to infinity in such a way that the position t remains finite. This implies that the corresponding lattice spacing τ cannot be modified by a point transformation but it is a fixed number which can tend to zero. So, for example, the lattice variable $t_{n,m}$ cannot be described by dilation invariant equations such as

$$t_{n+1,m} - 2t_{n,m} + t_{n-1,m} = 0 \qquad t_{n,m+1} - t_{n,m} = 0 \tag{2.23}$$

as in this case the parameter τ is an integration constant and not a parameter of the equations. Let us choose, consequently, the lattice equations for the variable $t_{n,m}$ as

$$t_{n+1,m} - t_{n,m} = \tau \qquad t_{n,m+1} - t_{n,m} = 0. \tag{2.24}$$

The solution of equation (2.24) reads

$$t \equiv t_{n,m} = \tau n + t_0 \tag{2.25}$$

where, in all generality, we can set $t_0 = 0$. As for the remaining lattice variable $x_{n,m}^{(2)} = x_{n,m}$, its position is not changing in time, i.e.

$$x_{n+1,m} - x_{n,m} = 0 \tag{2.26}$$

while its variation along the lattice, i.e. along m, can depend on t and on $u_{n,m}$ in a way that is defined by one of the equations (2.2), say $E_1 = 0$.

The continuous limit is obtained by considering $n \to \infty$ as $\tau \to 0$. In such a limit the variables $x_{n,m}$ and $u_{n,m}$ will no longer depend on n; equation (2.1), if it has the proper dependence on τ, will reduce to a differential-difference equation for $u_m = u(t, x_m)$ and equations (2.2) will reduce to just one equation for the lattice variable x_m, $E_1 = 0$, while the other equations (2.24), (2.26) are identically satisfied in the limit.

In this limit the symmetry vector $\hat{X}_{n,m}$, given by equation (2.5), reduces to $\hat{X}_m$, which will inherit the properties of $\hat{X}_{n,m}$ as applied to the lattice equations (2.24), (2.26) even if these equations in the limit reduce to $0 \equiv 0$. Applying equation (2.6) to equations (2.24), (2.26), with $x_{n,m}^{(1)} = t_{n,m}$ and $x_{n,m}^{(2)} = x_{n,m}$, we get the following three determining equations:

$$\xi^{(1)}(t_{n,m} + \tau, x_{n+1,m}, u_{n+1,m}) = \xi^{(1)}(t_{n,m}, x_{n,m}, u_{n,m}) \tag{2.27}$$

$$\xi^{(1)}(t_{n,m}, x_{n,m+1}, u_{n,m+1}) = \xi^{(1)}(t_{n,m}, x_{n,m}, u_{n,m}) \tag{2.28}$$

$$\xi^{(2)}(t_{n,m} + \tau, x_{n+1,m}, u_{n+1,m}) = \xi^{(2)}(t_{n,m}, x_{n,m}, u_{n,m}). \tag{2.29}$$

As the differential-difference equation will involve at least $u_{n+1,m}, u_{n,m}, u_{n,m+1}$, we can always take $u_{n,m}, u_{n,m+1}$ as independent variables and express $u_{n+1,m}$ in their terms. By differentiating equation (2.27) with respect to $u_{n,m+1}$ we get $\xi^{(1)}(t_{n,m}, x_{n,m})$. By a similar reasoning for the variable $x_{n,m}$ we can reduce the function $\xi^{(1)}$ to $\xi^{(1)}(t_{n,m})$, i.e. the function $\xi^{(1)}$ is just a function of t. In a similar way we will find that

$$\hat{X}_{n,m} = \xi^{(1)}(t_{n,m})\partial_{t_{n,m}} + \xi^{(2)}(t_{n,m}, x_{n,m})\partial_{x_{n,m}} + \phi(t_{n,m}, x_{n,m}, u_{n,m})\partial_{u_{n,m}} \tag{2.30}$$

and thus, in the continuum limit, we must have

$$\hat{X}_m = \xi^{(1)}(t)\partial_t + \xi^{(2)}(t, x_m)\partial_{x_m} + \phi(t, x_m, u_m)\partial_{u_m}. \tag{2.31}$$

3. The calculus of discrete symmetries

We will now discuss two methods of determining the discrete symmetries of differential equations. One of them was proposed by Hydon [7, 14], and it is essentially the classical method of constructing the normalizer of a group. The other, due to Gaeta and Rodríguez [8], is a modification of Lie's method, defining a discrete symmetry as a discretization of the parameter of a continuous symmetry. There are at least two other methods to construct discrete symmetries. One of them is based on a formulation of differential equations through differential forms [15] and the other [3–5] solves the determining equations for the group transformations directly. They will not be considered in this work.

3.1. Automorphisms of the symmetry algebra

Let us consider a differential equation for a dependent variable u and N independent variables $\mathbf{x} = (x_1, \ldots, x_N)$, with a Lie group of symmetries, G, and its corresponding Lie algebra $\mathcal{G}$.

We construct all the automorphisms of this Lie algebra. Some of them will, obviously, correspond to continuous symmetries. Others will be essentially new and will define discrete symmetries of the equation. And, finally, others will not be symmetries of the equation. These ideas are very well known in the theory of Lie algebras and groups. Inner automorphisms correspond to conjugation by elements of the group, while outer automorphisms have not this character. See, for instance, [6] where the complete and general Lorentz group are obtained in this way from the proper Lorentz group.

Let $\mathcal{G}$ be a Lie algebra of finite dimension n and $\mathcal{B} = \{X_1, \dots, X_n\}$ be a basis of $\mathcal{G}$. The commutation relations in this basis can be written as

$$[X_i, X_j] = c_{ij}^k X_k \tag{3.1}$$

where c_{ij}^k are the structure constants of $\mathcal{G}$ (a sum is understood over repeated indices running from 1 to n).

The defining equation for an automorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}$ is

$$\phi[X, Y] = [\phi(X), \phi(Y)] \tag{3.2}$$

which, in the basis $\mathcal{B}$, reads

$$\phi[X_i, X_j] = c_{ij}^k \phi(X_k) = [\phi(X_i), \phi(Y_j)]. \tag{3.3}$$

Let Φ be the matrix representation of ϕ in the basis $\mathcal{B}$ $(\det \Phi \neq 0)$. Then

$$\phi(X_i) = \Phi^j{}_i X_j \tag{3.4}$$

and equation (3.3) is written as

$$c_{ij}^k \Phi^l{}_k = \Phi^m{}_i \Phi^r{}_j c_{mr}^l \tag{3.5}$$

for all indices i, j, l (the equation is skewsymmetric in i, j). If we define the matrices of the adjoint representation

$$C(i)^k{}_j = c_{ij}^k \tag{3.6}$$

equation (3.5) can also be written as

$$\Phi C(i) = \Phi^l{}_i C(l) \Phi. \tag{3.7}$$

Automorphisms which are related through conjugation by an element of the symmetry group will be considered equivalent. Thus the matrix Φ can be simplified by conjugations with the symmetry transformations (at least if the algebra is not Abelian). This can be done in terms of the adjoint representation.

Let us consider those elements $g_i \in G$ which are generated by an element X_i of the basis $\mathcal{B}$ of the corresponding Lie algebra $\mathcal{G}$,

$$g_i = \mathrm{e}^{\lambda X_i}. \tag{3.8}$$

The conjugation provides the transformation

$$\phi \to \mathrm{e}^{-\lambda X_i} \phi \, \mathrm{e}^{\lambda X_i}. \tag{3.9}$$

We can consider the transformation of the elements of the basis $\mathcal{B}$ and we have

$$\mathrm{e}^{-\lambda X_i} X_j \, \mathrm{e}^{\lambda X_i} = A(i, \lambda)^k{}_j X_k. \tag{3.10}$$

It is not difficult to show that

$$A(i, \lambda) = \mathrm{e}^{\lambda C(i)}. \tag{3.11}$$

We can apply the transformations (3.11)

$$\Phi \to \mathrm{e}^{\lambda C(i)} \Phi \tag{3.12}$$

to simplify the matrix Φ.

We can now construct a representation of the automorphism in the space of variables. Let X be a vector field given by (the index a runs from 1 to N)

$$X = \xi^a(\mathbf{x}, u)\partial_{x_a} + \varphi(\mathbf{x}, u)\partial_u \tag{3.13}$$

which is a symmetry of the equation under study, and let us consider a symmetry, given by the transformation

$$(\mathbf{x}, u) \to (\hat{\mathbf{x}}(x, u), \hat{u}(x, u)). \tag{3.14}$$

The vector field X is transformed into a new vector field

$$\hat{X} = \hat{\xi}^a(\hat{\mathbf{x}}, \hat{u})\partial_{\hat{x}_a} + \hat{\varphi}(\hat{\mathbf{x}}, \hat{u})\partial_{\hat{u}}. \tag{3.15}$$

As the transformation is a symmetry of the equation, $\hat{X}$ must have the same form in the new variables,

$$\hat{X} = \xi^a(\hat{\mathbf{x}}, \hat{u})\partial_{\hat{x}_a} + \varphi(\hat{\mathbf{x}}, \hat{u})\partial_{\hat{u}}. \tag{3.16}$$

If we consider a basis of the symmetry algebra, $\{X_i\}$, $i = 1, \ldots, n$, the transformed vector fields are

$$\hat{X}_i = \xi_i^a(\hat{\mathbf{x}}, \hat{u})\partial_{\hat{x}_a} + \varphi_i(\hat{\mathbf{x}}, \hat{u})\partial_{\hat{u}} \tag{3.17}$$

and, as the transformation is an automorphism of the algebra,

$$\hat{X}_i = \Phi^j{}_i X_j. \tag{3.18}$$

Applying (3.18) to the new variables, we get

$$\hat{X}_i\hat{x}_a = \xi_i^a(\hat{\mathbf{x}}, \hat{u}) = \Phi^j{}_i X_j\hat{x}_a = \Phi^j{}_i\left(\xi_j^b(\mathbf{x}, u)\frac{\partial\hat{x}_a}{\partial x_b} + \varphi_j(\mathbf{x}, u)\frac{\partial\hat{x}_a}{\partial u}\right) \tag{3.19}$$

and

$$\hat{X}_i\hat{u} = \Phi_i(\hat{\mathbf{x}}, \hat{u}) = \Phi^j{}_i X_j\hat{u} = \Phi^j{}_i\left(\xi_j^b(\mathbf{x}, u)\frac{\partial\hat{u}}{\partial x_b} + \varphi_j(\mathbf{x}, u)\frac{\partial\hat{u}}{\partial u}\right) \tag{3.20}$$

or

$$\begin{aligned}
&\xi_j^b(\mathbf{x}, u)\frac{\partial\hat{x}_a}{\partial x_b} + \varphi_j(\mathbf{x}, u)\frac{\partial\hat{x}_a}{\partial u} = (\Phi^{-1})^i{}_j\xi_i^a(\hat{\mathbf{x}}, \hat{u})\\
&\xi_j^b(\mathbf{x}, u)\frac{\partial\hat{u}}{\partial x_b} + \varphi_j(\mathbf{x}, u)\frac{\partial\hat{u}}{\partial u} = (\Phi^{-1})^i{}_j\varphi_i(\hat{\mathbf{x}}, \hat{u}).
\end{aligned} \tag{3.21}$$

We have to solve equations (3.21) to find the expression of the automorphism as a transformation in our space of variables and functions. After, we must check if the automorphism is a symmetry of the equation and that it does not correspond to a continuous symmetry.

3.2. Determining the equation for a discrete symmetry

We will now consider the method of the discretization of the parameter of a continuous transformation. Let us briefly review how the method works. To simplify the description (see [8] for a detailed exposition), let us consider a smooth curve in $\mathbf{R}^2$, $y = f(x)$ and a vector field

$$X = \xi(x, y)\partial_x + \varphi(x, y)\partial_y. \tag{3.22}$$

The point (x, y) is transformed under the infinitesimal action as

$$x' = x + \lambda\xi(x, y) \qquad y' = y + \lambda\varphi(x, y) \tag{3.23}$$

and the graph of our curve is transformed into a new one, defined by the transformed function

$$f_\lambda(x) = f(x) + \lambda[\varphi(x, y) - \xi(x, y)\partial_x f(x)]. \tag{3.24}$$

If we introduce a function $F(x; \lambda)$ such that $F(x; \lambda) = f_\lambda(x)$, F, taking into account (3.24), will satisfy the partial differential equation

$$\frac{\partial F(x; \lambda)}{\partial \lambda} + \xi(x, F(x; \lambda))\frac{\partial F(x; \lambda)}{\partial x} = \varphi(x, F(x; \lambda)). \tag{3.25}$$

In order to recover the original function, we have to impose the initial condition

$$F(x; 0) = f(x) \tag{3.26}$$

and if we want to obtain the same function for a particular value λ_0 of the parameter and hence a discrete symmetry

$$F(x; \lambda_0) = F(x; 0) = f(x). \tag{3.27}$$

This is equivalent to finding periodic solutions of equation (3.25). This equation is a functional equation and hard to solve.

If we consider a differential equation instead of the graph of a function, we have to pose the same question in the appropriate jet space [1]. The answer is simpler in this case, as the right-hand side of equation (3.25) is computed from the prolongation of the vector field under consideration.

Let us consider a differential equation,

$$\partial_J u = f(\mathbf{x}, u, \partial_{J'} u, \ldots) \tag{3.28}$$

where $J = (j_1, \ldots, j_N)$ and $\partial_J u = \partial_{x_1}^{j_1} \cdots \partial_{x_N}^{j_N} u$, $J = (j_1, \ldots, j_N)$.

The determining equation, used to describe the discrete symmetries, is then

$$\frac{\partial F}{\partial \lambda} + \sum_i \xi_i \frac{\partial F}{\partial x_i} + \sum_{J'} \phi^{J'} \frac{\partial F}{\partial u_{J'}} = \phi^J \bigg|_{\partial_J u = F} \tag{3.29}$$

where $\phi^{J'}$ are the J'-prolongations of the vector field (3.13) and one has to rewrite equation (3.28) in terms of F. Then one looks for periodic solutions of this equation in λ.

For instance, consider a differential equation of the following type

$$u_{tt} = f(\mathbf{x}, u, u_t). \tag{3.30}$$

Choosing $F = F(\mathbf{x}, u, u_t; \lambda) = u_{tt}$, the determining equation is

$$\frac{\partial F}{\partial \lambda} + \sum_i \xi_i \frac{\partial F}{\partial x_i} + \phi \frac{\partial F}{\partial u} + \phi^t \frac{\partial F}{\partial u_t} = \phi^{tt} \bigg|_{\partial_{tt} u = F} \tag{3.31}$$

and

$$F(\mathbf{x}, u, u_t; 0) = F(\mathbf{x}, u, u_t; \lambda_0) = f(\mathbf{x}, u, u_t) \tag{3.32}$$

for some λ_0.

To apply this method we do not need to know the continuous symmetries. In fact, the determining equation provides a general solution which does not depend on the differential equation we are studying. The equation appears when we impose the boundary condition as in (3.26) and the discrete symmetries by requiring equation (3.27). However, we have to make some assumptions to get a solution of the equation, because we do not know the symmetries which appear in it. We will use this method to determine the discrete symmetries of the Painlevé I equation (2.8).

4. Lie discrete symmetries of lattice equations

The construction (shown in section 2) of point symmetries of discrete equations is very similar to the standard approach to point symmetries of continuous equations [1]. The main difference lies in the form of the prolongation and in the way we solve the determining equations. Consequently, the procedure presented in section 3 for constructing discrete symmetries of continuous equations can be carried over in a straightforward manner to the discrete case, just by changing the form of the prolongation. So, in the following we will just apply the methods discussed in section 3 to two examples of equations of mathematical physics, the discrete Painlevé equation (2.8) and the Toda equation [16]. The Painlevé equation, as we have shown in section 2, has no continuous symmetry (for a generic choice of the parameters), and we will use the determining equation method to find its discrete symmetries. In the Toda equation we have a continuous group of symmetries [17, 18], and we will construct the discrete symmetries by computing the automorphisms of the corresponding Lie algebra.

4.1. Discrete Painlevé equation

As we have said in the introduction to this section, the discrete Painlevé I equation (2.8) has no continuous symmetry so that we cannot apply the method of the normalizer.

We define the extended λ-dependent equation in the transformed space as

$$\hat{u}_+ + \hat{u}_- = F(\hat{x}, \hat{u}; \lambda) \tag{4.1}$$

where, for $\lambda = 0$

$$F(\hat{x}, \hat{u}; 0) = -u + \frac{\alpha x + \beta}{u} + \gamma. \tag{4.2}$$

For the sake of simplicity, we have used the notation

$$x = x_n \qquad x_+ = x_{n+1} \qquad x_- = x_{n-1}$$
$$u = u(x_n) \qquad u_+ = u(x_{n+1}) \qquad u_- = u(x_{n-1}).$$

The generic Lie point transformation, written in terms of the infinitesimal symmetry generators, is

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\lambda} = \xi(\hat{x}, \hat{u}) \qquad \frac{\mathrm{d}\hat{u}}{\mathrm{d}\lambda} = \phi(\hat{x}, \hat{u}). \tag{4.3}$$

We consider the same definition of the lattice (2.9) that we used for constructing the Lie point symmetries in section 2. So we have $\xi = K_0$. Differentiating equation (4.1) with respect to λ and taking into account equation (4.3) we get (3.31)

$$\frac{\partial F}{\partial \lambda} + K_0 \frac{\partial F}{\partial \hat{x}} + \phi(\hat{x}, \hat{u}) \frac{\partial F}{\partial \hat{u}} = \phi(\hat{x}_+, \hat{u}_+) + \phi(\hat{x}_-, F - \hat{u}_+). \tag{4.4}$$

By differentiating equation (4.4) with respect to $\hat{u}_+$ and $\hat{u}$ we get

$$\phi(\hat{x}, \hat{u}) = \phi^{(0)}(\hat{x}) + \hat{u}\phi^{(1)}(\hat{x}). \tag{4.5}$$

Substituting equation (4.5) into equation (4.4) and requiring that the obtained equation be satisfied for any u_+, we get

$$\phi^{(1)}(x) = K_1 + K_2(-1)^{\frac{\hat{x}-\hat{x}_0}{h}}. \tag{4.6}$$

Equation (4.4) is now reduced to a linear partial differential equation of first order which can be solved on the characteristics:

$$\frac{\mathrm{d}\lambda}{1} = \frac{\mathrm{d}\hat{x}}{K_0} = \frac{\mathrm{d}\hat{u}}{\phi^{(0)} + \hat{u}\phi^{(1)}} = \frac{\mathrm{d}F}{\phi_+^{(0)} + \phi_-^{(0)} + \phi_-^{(1)} F} \tag{4.7}$$

where

$$\phi^{(i)} = \phi^{(i)}(x) \qquad \phi_+^{(i)} = \phi^{(i)}(x_+) \qquad \phi_-^{(i)} = \phi^{(i)}(x_-) \qquad i = 0, 1. \tag{4.8}$$

The first invariant is

$$\hat{x} = x + K_0\lambda. \tag{4.9}$$

The action of this transformation on the discrete Painlevé equation (2.8) corresponds just to a change in the parameter β. We could carry out the calculation with $K_0 \neq 0$, but for the sake of clarity of the presentation, we have set $K_0 = 0$ as this transformation cannot provide any discrete Lie transformation. Consequently, $\phi^{(i)}$ will not depend on λ.

The next invariant is obtained by integrating equation (4.7) for $\hat{u}$ as a function of λ. We have

$$\hat{u} = \mathrm{e}^{\lambda\phi^{(1)}} \left[u + \frac{\phi^{(0)}}{\phi^{(1)}} \left(1 - \mathrm{e}^{-\lambda\phi^{(1)}}\right) \right]. \tag{4.10}$$

The integration of equation (4.7) for F satisfying the boundary conditions (4.2) gives

$$\begin{aligned} F(\hat{x}, \hat{u}; \lambda) = {} & \mathrm{e}^{\lambda\phi_-^{(1)}} \left[\frac{\phi^{(0)}}{\phi^{(1)}} \left(1 - \mathrm{e}^{-\lambda\phi^{(1)}}\right) + \frac{\alpha\hat{x} + \beta}{\hat{u}\,\mathrm{e}^{-\lambda\phi^{(1)}} - \frac{\phi^{(0)}}{\phi^{(1)}}\left(1 - \mathrm{e}^{-\lambda\phi^{(1)}}\right)} + \gamma \right] \\ & - \hat{u}\,\mathrm{e}^{\lambda[\phi_-^{(1)} - \phi^{(1)}]} - \frac{\phi_+^{(0)} + \phi_-^{(0)}}{\phi_-^{(1)}} \left(1 - \mathrm{e}^{\lambda\phi_-^{(1)}}\right). \end{aligned} \tag{4.11}$$

To get a discrete Lie symmetry we require that there exists a value of λ, say λ_0, such that

$$F(\hat{x}, \hat{u}; 0) = F(\hat{x}, \hat{u}; \lambda_0). \tag{4.12}$$

If we want (4.12) to be satisfied, we need

$$(\phi_-^{(1)} - \phi^{(1)})\lambda_0 = 2\pi\mathrm{i}N \qquad N \in \mathbf{Z} \tag{4.13}$$

$$\phi^{(0)}\left(1 - \mathrm{e}^{\lambda_0\phi^{(1)}}\right) = 0. \tag{4.14}$$

From (4.13) we obtain $K_2 = 0$ and then, $\phi^{(1)} = K_1$. Equation (4.14) is solved by requiring one or the other of the following two conditions:

$$\lambda_0 K_1 = 2\pi\mathrm{i}N \qquad N \in \mathbf{Z} \tag{4.15}$$

$$\phi^{(0)} = 0. \tag{4.16}$$

In the case of condition (4.15), equation (4.12) is satisfied. However, this provides no discrete symmetry.

Let us go over to the second condition (4.16). If $\gamma \neq 0$, equation (4.12) implies equation (4.15), i.e. no discrete symmetry is present. If $\gamma = 0$, then $K_1\lambda_0 = \mathrm{i}\pi N$, providing a discrete symmetry $\hat{u} = \pm u$ even in the case when no continuous symmetry is present. However, in this case, the continuum limit of this difference equation is not Painlevé I.

4.2. Discrete symmetries of the Toda equation

Let us consider as our second example the Toda equation:

$$u_{tt} = \mathrm{e}^{u_+ - u} - \mathrm{e}^{u - u_-} \tag{4.17}$$

where $x_\pm = x \pm h$, $u_\pm = u(x \pm h, t)$ and h is the lattice step (see equation (2.9)).

As is well known [17–19], the following operators form a basis of the symmetry algebra of the Toda equation:

$$X_1 = \partial_u \qquad X_2 = \partial_x \qquad X_3 = \partial_t \qquad X_4 = t\partial_u \qquad X_5 = t\partial_t - \frac{2x}{h}\partial_u. \tag{4.18}$$

The nonzero commutation relations are

$$[X_2, X_5] = -\frac{2}{h}X_1 \qquad [X_3, X_4] = X_1 \qquad [X_3, X_5] = X_3 \qquad [X_4, X_5] = -X_4 \tag{4.19}$$

and the matrices $C(i)$ of the adjoint representation are

$$C(1) = 0 \qquad C(2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -2/h \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad C(3) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$
$$C(4) = \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad C(5) = \begin{pmatrix} \cdot & 2/h & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{4.20}$$

Applying equation (3.7) we get

$$b_{21} = b_{23} = b_{24} = b_{31} = b_{41} = b_{51} = b_{53} = b_{54} = 0 \tag{4.21}$$

where, to simplify the notation we have written $\Phi^i{}_j = b_{ij}$. The most significant remaining equations are

$$b_{34}(b_{55} + 1) = b_{44}(b_{55} - 1) = b_{33}(b_{55} - 1) = b_{43}(b_{55} + 1) = 0 \tag{4.22}$$

$$b_{11} = b_{33}b_{44} - b_{34}b_{43}. \tag{4.23}$$

The determinant of Φ must be different from zero (ϕ is an automorphism) and consequently, taking into account equation (4.21), we must have $b_{11} \neq 0$. Using equation (4.22) we conclude that $b_{55} = \pm 1$. We will distinguish two cases:

(a) $b_{55} = 1$. In this case, $b_{34} = b_{43} = 0$ and $b_{33} \neq 0, b_{44} \neq 0$. The other elements of the matrix Φ must satisfy the equations

$$\begin{aligned} &b_{32} = b_{42} = b_{52} = 0 \qquad && b_{13} = b_{33}b_{45} \\ &b_{14} = b_{35}b_{44} && b_{11} = b_{22} = b_{33}b_{44}. \end{aligned} \tag{4.24}$$

The matrix $\Phi_1 = \Phi(b_{55} = 1)$ is

$$\Phi_1 = \begin{pmatrix} b_{33}b_{44} & b_{12} & b_{33}b_{45} & b_{35}b_{44} & b_{15} \\ 0 & b_{33}b_{44} & 0 & 0 & b_{25} \\ 0 & 0 & b_{33} & 0 & b_{35} \\ 0 & 0 & 0 & b_{44} & b_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.25}$$

(b) $b_{55} = -1$. Now, $b_{33} = b_{44} = 0$ and $b_{34} \neq 0, b_{43} \neq 0$. The other elements satisfy the equations

$$\begin{aligned} &b_{32} = b_{42} = b_{52} = 0 \qquad && b_{13} = -b_{35}b_{43} \\ &b_{14} = -b_{34}b_{45} && b_{11} = -b_{22} = -b_{34}b_{43}. \end{aligned} \tag{4.26}$$

The matrix $\Phi_2 = \Phi(b_{55} = -1)$ is

$$\Phi_2 = \begin{pmatrix} -b_{34}b_{43} & b_{12} & -b_{35}b_{43} & -b_{34}b_{45} & b_{15} \\ 0 & b_{34}b_{43} & 0 & 0 & b_{25} \\ 0 & 0 & 0 & b_{34} & b_{35} \\ 0 & 0 & b_{43} & 0 & b_{45} \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.27}$$

To simplify the matrix Φ we will conjugate the automorphism using the continuous transformations in the adjoint representation.

The exponentials of the matrices $C(i)$ are easy to find. Using $C(3)$ we can put $b_{35} = 0$, with $C(4)$, $b_{45} = 0$, $C(5)$ gives $b_{12} = 0$, and, finally using $C(2)$, $b_{15} = 0$. Then, the simplified Φ_1 is

$$\Phi_1 = \begin{pmatrix} b_{33}b_{44} & 0 & 0 & 0 & 0 \\ 0 & b_{33}b_{44} & 0 & 0 & b_{25} \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad b_{33}, b_{44} \neq 0. \tag{4.28}$$

The same procedure can be used with Φ_2 and the result is

$$\Phi_2 = \begin{pmatrix} -b_{34}b_{43} & 0 & 0 & 0 & 0 \\ 0 & b_{34}b_{43} & 0 & 0 & b_{25} \\ 0 & 0 & 0 & b_{34} & 0 \\ 0 & 0 & b_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad b_{34}, b_{43} \neq 0. \tag{4.29}$$

The following step is to realize the automorphisms in the space of the variables and functions of the Toda equation. We have to solve the following system of equations:

$$\begin{aligned} \tau_j(t,x,u)\frac{\partial \hat{t}}{\partial t} + \xi_j(t,x,u)\frac{\partial \hat{t}}{\partial x} + \varphi_j(t,x,u)\frac{\partial \hat{t}}{\partial u} &= (\Phi^{-1})^i{}_j \tau_i(\hat{t},\hat{x},\hat{u}) \\ \tau_j(t,x,u)\frac{\partial \hat{x}}{\partial t} + \xi_j(t,x,u)\frac{\partial \hat{x}}{\partial x} + \varphi_j(t,x,u)\frac{\partial \hat{x}}{\partial u} &= (\Phi^{-1})^i{}_j \xi_i(\hat{t},\hat{x},\hat{u}) \\ \tau_j(t,x,u)\frac{\partial \hat{u}}{\partial t} + \xi_j(t,x,u)\frac{\partial \hat{u}}{\partial x} + \varphi_j(t,x,u)\frac{\partial \hat{u}}{\partial u} &= (\Phi^{-1})^i{}_j \varphi_i(\hat{t},\hat{x},\hat{u}) \end{aligned} \tag{4.30}$$

where $j = 1, \ldots, 5$. In case (a),

$$\Phi_1^{-1} = \begin{pmatrix} \mu\nu & 0 & 0 & 0 & 0 \\ 0 & \mu\nu & 0 & 0 & \sigma \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mu, \nu \neq 0. \tag{4.31}$$

Then, for $j = 1$, $X_1 = \partial_u$

$$\frac{\partial \hat{t}}{\partial u} = 0 \qquad \frac{\partial \hat{x}}{\partial u} = 0 \qquad \frac{\partial \hat{u}}{\partial u} = \mu\nu. \tag{4.32}$$

For $j = 2$, $X_2 = \partial_x$

$$\frac{\partial \hat{t}}{\partial x} = 0 \qquad \frac{\partial \hat{x}}{\partial x} = \mu\nu \qquad \frac{\partial \hat{u}}{\partial x} = 0. \tag{4.33}$$

For $j = 3$, $X_3 = \partial_t$

$$\frac{\partial \hat{t}}{\partial t} = \mu \qquad \frac{\partial \hat{x}}{\partial t} = 0 \qquad \frac{\partial \hat{u}}{\partial t} = 0. \tag{4.34}$$

For $j = 4$, $X_4 = t\partial_u$

$$t\frac{\partial \hat{t}}{\partial u} = 0 \qquad t\frac{\partial \hat{x}}{\partial u} = 0 \qquad t\frac{\partial \hat{u}}{\partial u} = \nu \hat{t}. \tag{4.35}$$

For $j = 5$, $X_5 = t\partial_t - \frac{2x}{h}\partial_u$

$$t\frac{\partial \hat{t}}{\partial t} - \frac{2x}{h}\frac{\partial \hat{t}}{\partial u} = \hat{t} \qquad t\frac{\partial \hat{x}}{\partial t} - \frac{2x}{h}\frac{\partial \hat{x}}{\partial u} = \sigma \qquad t\frac{\partial \hat{u}}{\partial t} - \frac{2x}{h}\frac{\partial \hat{u}}{\partial u} = -\frac{2\hat{x}}{h}. \tag{4.36}$$

The solution of equations (4.32), (4.33) and (4.34) is

$$\hat{t} = \mu t + \alpha \qquad \hat{x} = \mu\nu x + \beta \qquad \hat{u} = \mu\nu u + \gamma. \tag{4.37}$$

Substituting in (4.35) we get $\alpha = 0$ and in (4.36) we get $\sigma = 0$ and $\beta = 0$. The translation in u is a continuous symmetry so the possible discrete symmetries are

$$\hat{t} = c_1 t \qquad \hat{x} = c_2 x \qquad \hat{u} = c_2 u \qquad c_1, c_2 \neq 0. \tag{4.38}$$

Finally, we check if this transformation is a symmetry of the equation. To do so, we apply it to the Toda equation (4.17). The Toda equation in the new variables reads

$$\frac{c_1^2}{c_2}\hat{u}_{\hat{t}\hat{t}} = \exp\left(\frac{1}{c_2}\left[\hat{u}\left(\hat{t}, \hat{x} + \frac{\hat{h}}{c_2}\right) - \hat{u}(\hat{t}, \hat{x})\right]\right) - \exp\left(\frac{1}{c_2}\left[\hat{u}(\hat{t}, \hat{x}) - \hat{u}\left(\hat{t}, \hat{x} - \frac{\hat{h}}{c_2}\right)\right]\right) \tag{4.39}$$

with $\hat{h} = c_2 h$. Then, $c_2 = \pm 1, c_1 = \pm 1$ are the only admissible solutions. When $c_2 = 1, c_1 = -1$, we obtain a discrete symmetry, given by the transformation:

$$\hat{t} = -t \qquad \hat{x} = x \qquad \hat{u} = u. \tag{4.40}$$

When $c_2 = -1, c_1 = 1$, we get another discrete symmetry:

$$\hat{t} = t \qquad \hat{x} = -x \qquad \hat{u} = -u. \tag{4.41}$$

It is easy to check that the automorphism of case (b) cannot be realized in the representation under consideration.

If we require that the discrete symmetries be preserved under the transformation which carries the Toda lattice into the Korteweg–de Vries equation [18, 19], we get that $c_2 = c_1$. So only one symmetry is preserved in the continuous limit.

The existence of discrete symmetries is associated with the existence of admissible boundary conditions. In the case of the Toda lattice, the discrete symmetry (4.40) allows us to construct solutions invariant under time inversion while the existence of solutions symmetric with respect to the origin is due to the symmetry (4.41).

5. Conclusions

In this paper we have shown that the two methods, the automorphisms of the symmetry algebra [14] and the determining equation for discrete symmetries [8], provide discrete symmetries, even in the case of discrete equations. To verify the theory presented in this paper many problems have been considered. We have just shown here the more significant examples.

There are some drawbacks for both methods.

- In the case of the automorphism method, if the symmetry group is very large, the matrices involved become big and the final defining equations are very overdetermined and, in some cases, may need symbolic manipulation programmes to carry out the calculations. Moreover, the method is not applicable if there are no continuous symmetries, as is the case, for example, for the Painlevé equations. It is worthwhile to note that in the case of the Toda lattice, we have obtained a non-obvious discrete symmetry using these techniques.
- In the case of the determining equation method, the equation for the function F can sometimes be undetermined. For example, in the case of the Volterra equation

$$u_t = u(u_+ - u_-) \tag{5.1}$$

having chosen $F = F(t, u, u_+, u_-; \lambda) = u_t$ the determining equation to solve is

$$\frac{\partial F}{\partial \lambda} + \tau(t)\frac{\partial F}{\partial t} + \phi(t,u)\frac{\partial F}{\partial u} + \phi(t,u_+)\frac{\partial F}{\partial u_+} + \phi(t,u_-)\frac{\partial F}{\partial u_-} = \phi_t + (\phi_u - \tau')F. \tag{5.2}$$

We have no hint of the form of $\phi(t, u)$ and, consequently, equation (5.2) is not solvable. Of course, one could try to introduce an ansatz for this function and may find some discrete symmetries. However, as one can easily show using the automorphism method, the Volterra equation has no discrete symmetries and an ansatz will not provide any conclusion.

Work is in progress to apply the other techniques mentioned at the beginning of section 3 and for the construction of lattices and discrete equations with prescribed discrete symmetries, a problem of interest in chemistry and quantum mechanics. Another topic which attracts our attention is the study of solutions of discrete equations invariant under a discrete symmetry group.

Acknowledgments

This work was done while DL was visiting the Departamento de Física Teórica II of Universidad Complutense de Madrid (Spain). His visit was financed by the *Ministerio de Educación, Cultura y Deportes* of Spain (SAB2001-0140). This work was also partially supported by Ministry of Science and Technology (Spain) under grant BFM2002-02646 and also by NATO under grant PST.CLG.978431.

References

[1] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[2] Levi D and Winternitz P 1993 Symmetries and conditional symmetries of differential difference equations *J. Math. Phys.* **34** 3713–30
[3] Reid G J, Weih D T and Wittkopf A D 1993 A point symmetry group of a differential equation which cannot be found using infinitesimal methods *Modern Group Analysis: Advances Analytical and Computational Methods in Mathematical Physics* ed N H Ibraghimov, M Torrisi and A Valenti (Dordrecht: Kluwer) pp 93–9
[4] Lisle I G and Reid G J 1998 Geometry and structure of Lie pseudogroups from infinitesimal defining systems *J. Symb. Comput.* **26** 355–79
[5] Reid G J, Boulton A and Lisle I G 1999 Characterizing Lie equations by their infinitesimal symmetries unpublished
[6] Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Groups and Their Applications* (London: Pergamon)
[7] Hydon P E 1998 Discrete point symmetries of ordinary differential equations *Proc. R. Soc.* A **454** 1961–72

[8] Gaeta G and Rodríguez M A 1996 Discrete symmetries of differential equations *J. Phys. A: Math. Gen.* **29** 859–80

[9] Levi D, Tremblay S and Winternitz P 2001 Lie point symmetries of difference equations and lattices *J. Phys. A: Math. Gen.* **33** 521–54

[10] Levi D, Tremblay S and Winternitz P 2001 Lie symmetries of multidimensional difference equations *J. Phys. A: Math. Gen.* **34** 9507–24

[11] Levi D, Ragnisco O and Rodríguez M A 1992 On non-isospectral flows, Painlevé equations and symmetries of differential and difference equations *Teor. Mat. Fiz.* **93** 473–80

[12] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)

[13] Kelley M G and Peterson A C 1991 *Difference Equations: An Introduction with Applications* (San Diego, CA: Academic)

[14] Hydon P E 2000 How to construct the discrete symmetries of partial differential equations *Eur. J. Appl. Math.* **11** 515–27

[15] Kistovich A V and Chaschechkin Yu D 2001 Regular method for searching of differential equations discrete symmetries *Regul. Chaotic Dyn.* **6** 327–36

[16] Toda M 1981 *Theory of Nonlinear Lattices* (Berlin: Springer)

[17] Levi D and Winternitz P 1991 Continuous symmetries of discrete equations *Phys. Lett.* A **152** 335–8

[18] Hernández Heredero R, Levi D, Rodríguez M A and Winternitz P 2000 Lie algebra contractions and symmetries of the Toda hierarchy *J. Phys. A: Math. Gen.* **33** 5025–40

[19] Levi D and Rodríguez M A 1992 Symmetry group of partial differential equations and of differential difference equations: the Toda lattice versus the Korteweg–de Vries equation *J. Phys. A: Math. Gen.* **25** L975–9